

New Versions of the All-Ones Problem ^{*}

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Abstract

We study three new versions of the All-Ones Problem and the Minimum All-Ones Problem. The original All-Ones Problem is simply called the Vertex-Vertex Problem, and the three new versions are called the Vertex-Edge Problem, the Edge-Vertex Problem and the Edge-Edge Problem, respectively. The Vertex-Vertex Problem has been studied extensively. For example, existence of solutions and efficient algorithms for finding solutions were obtained, and the Minimum Vertex-Vertex Problem for general graphs was shown to be NP-complete and for trees it can be solved in linear time, etc. In this paper, for the Vertex-Edge Problem, we show that a graph has a solution if and only if it is bipartite, and therefore it has only two possible solutions and optimal solutions. A linear program version is also given. For the Edge-Vertex Problem, we show that a graph has a solution if and only if it contains even number of vertices. By showing that the Minimum Edge-Vertex Problem can be polynomially transformed into the Minimum Weight Perfect Matching Problem, we obtain that the Minimum Edge-Vertex Problem can be solved in polynomial time in general. The Edge-Edge Problem is reduced to the Vertex-Vertex Problem for the line graph of a graph.

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1 Introduction

The term *All-Ones Problem* was introduced by Sutner, see [12]. It has applications in linear cellular automata, see [13] and the references therein. The problem is cited as follows: suppose each of the square of an $n \times n$ chessboard is equipped with an indicator light and a button. If the button of a square is pressed, the light of that square will change from off to on and vice versa; the same happens to the lights of all the edge-adjacent squares. Initially all lights are off. Now, consider the following questions: is it possible to press a sequence of buttons in such a way that in the end all lights are on ? This is referred as the *All-Ones Problem*. If there is such a solution, how to find a such way ? And finally, how to find such a way that presses as few buttons as possible ? This is referred as the *Minimum All-Ones Problem*. All the above questions can be asked for arbitrary graphs. Here and in what follows, we consider connected simple undirected graphs only. One can deal with disconnected graphs component by component. For all

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terminology and notations on graphs, we refer to [2]. An equivalent version of the All-Ones Problem was proposed by Peled in [10], where it was called the *Lamp Lighting Problem*. The rule of the All-Ones Problem is called σ^+ rule on graphs, which means that a button lights not only its neighbors but also its own light. If a button lights only its neighbors but not its own light, this rule on graphs is called σ rule.

There have been many publications on the All-Ones Problem, see Sutner [14,15], Barua et al [1] and Dodis and Winkler [4]. Using linear algebra, Sutner [13] proved that it is always possible to light every lamp in any graphs by σ^+ rule. Lossers [7] gave another beautiful proof also by using linear algebra. A graph-theoretic proof was given by Eriksson et al [6]. In [3] we gave a graph-theoretic algorithm of cubic time for finding the solutions. In [11], Sutner proved that the Minimum All-Ones Problem is NP-complete in general. We gave [3] a linear time algorithm for finding optimal solutions for trees.

In graph-theoretic terminology, a solution to the All-Ones Problem with σ^+ -rule can be stated as follows: given a graph $G = (V, E)$, where V and E denotes the vertex-set and the edge-set of G , respectively. A subset X of V is a solution if and only if for every vertex v of G the number of vertices in X adjacent to or equal to v is odd. Such a subset X is called an *odd parity cover* in [12]. So, the All-Ones Problem can be formulated as follows: given a graph $G = (V, E)$, does a subset X of V exist such that for all vertex $v \in V - X$, the number of vertices in X adjacent to v is odd, while for all vertex $v \in X$, the number of vertices in X adjacent to v is even ? If there exists a solution, how to find a one with minimum cardinality ? We simply call them the *Vertex-Vertex Problem*, in contrasting with the following new versions of the All-Ones Problem.

From the Vertex-Vertex Problem, one is easily led to propose the following three problems. These problems have their own interests and are worth to be studied from algorithmic point of view, even if not related to the All-Ones Problem.

The *Vertex-Edge Problem*: suppose there is a light bulb on each edge and a button on each vertex. If we press the button on a vertex, the on/off status of the lights on the edges incident with the vertex will be changed. If the initial status of all lights are off, we ask whether one can press some buttons to light all the lights on the edges. This problem can be formulated in graph-theoretic terminology as follows: given a graph $G = (V, E)$, does there exist a subset X of V such that for all edge $e \in E$, exact one of its end-vertices is in X ? If there exists a solution, how find a one with minimum cardinality ?

The *Edge-Vertex Problem*: suppose there is a light bulb on each vertex and a button on each edge. If we press the button on an edge, the on/off status of the lights on the vertices incident with the edge will be changed. If the initial status of all lights are off, we ask whether one can press some buttons to light all the lights on the vertices. This problem can be formulated in graph-theoretic terminology as follows: given a graph $G = (V, E)$, does there exist a subset X of E such that for all vertex $v \in V$, the number of edges in X incident with v is odd ? If there exists a solution, how to find a one with minimum cardinality ?

The *Edge-Edge Problem*: suppose there is a light bulb and a button on each edge. If we press the button on an edge, the on/off status of the light on this edge and those on the edges adjacent to the edge will be changed. If the initial status of all lights are off, we ask

whether one can press some buttons to light all the lights on the edges. This problem can also be formulated in graph-theoretic terminology as follows: given a graph $G = (V, E)$, does there exist a subset F of E such that for all edge $e \in E - F$, the number of edges in F adjacent to e is odd, while for all edge $e \in F$, the number of edges in F adjacent to e is even? If there exist a solution, how to find a one with minimum cardinality? It is easily seen that this problem is equivalent to the Vertex-Vertex Problem for the line graph of a graph.

This paper is organized as follows. Section 1 is an introduction. In Section 2, we show that a graph has a solution to the Vertex-Edge Problem if and only if it is bipartite, and therefore it has only two possible solutions and optimal solutions. A linear programming version is also given. In Section 3, we show that a graph has a solution to the Edge-Vertex Problem if and only if it contains even number of vertices. By showing that the Minimum Edge-Vertex Problem can be polynomially transformed into the Minimum Weight Perfect Matching Problem, we obtain that the Minimum Edge-Vertex Problem can be solved in polynomial time in general. As we already pointed out, in Section 4, the Edge-Edge Problem is simply reduced to the Vertex-Vertex Problem for the line graph of a graph, and therefore there always exist solutions for general graphs, and a solution can be found in polynomial time.

2 The Vertex-Edge Problem

The Vertex-Edge Problem and its corresponding minimum problem can be completely solved, which is the easiest problem among the four problems. In this section we will find all the graphs that have a solution for the Vertex-Edge Problem, and figure out the two possible solutions.

Theorem 2.1. The Vertex-Edge Problem has a solution for a graph G if and only if G is bipartite. Moreover, a connected bipartite graph $G = (U, V, E)$ has only two solutions U and V . Therefore, the solution to the Minimum Vertex-Edge Problem is U or V such that it attains the value $\min\{|U|, |V|\}$.

Proof. For the first statement of the theorem, if G is bipartite, say $G = (U, V, E)$, then it is obvious that U and V are solutions to the Vertex-Edge Problem for G . Conversely, if G has a solution for the Vertex-Edge Problem, we only need to show that there does not exist odd cycle in G . Otherwise, suppose C is an odd cycle in G . Then, the lights on the edges of C should be on in the end. Because all the vertices incident to the edges of C are on the cycle, there must be a way to press some of the buttons on the vertices of the cycle C to light all the lights on the edges of C , which is impossible since C is an odd cycle, a contradiction. Therefore, G does not have odd cycles, and so G is bipartite.

For the second statement of the theorem, suppose that $G = (U, V, E)$ is a connected bipartite graph, and $U_1 \cup V_2$ is a solution to the Vertex-Edge Problem for G such that $U \supseteq U_1 \neq \emptyset, V \supseteq V_2 \neq \emptyset$. Consider the subgraph (U_1, V_1) of G induced by U_1 and the incident edges, and the subgraph (U_2, V_2) of G induced by V_2 and the incident edges. We claim that in G there is no edge between the subgraph (U_1, V_1) and the subgraph (U_2, V_2) . Otherwise, suppose that there is an edge e between U_1 and V_2 , then e can not be lighted because the two end-vertices of e are in the solution. Suppose e is between U_2 and V_1 ,

then e can not be lighted because both of the two end-vertices of e are not in the solution. In any case, the edge e can not be lighted by the solution $U_1 \cup V_2$, a contradiction. So, (U_1, V_1) is not connected with (U_2, V_2) , a contradiction to the assumption that G is connected. Therefore, one of U_1 and V_2 must be empty. If $U_1 = \emptyset$, then $V_2 = V$ since $U_1 \cup V_2$ is a solution, and if $V_2 = \emptyset$, then $U_1 = U$ by the same argument. Hence, G has only two solutions for the Vertex-Edge Problem.

The third statement of the theorem for the Minimum Vertex-Edge Problem is obvious. The proof is now complete. ■

For a disconnected bipartite graph, we can find the solutions for its every connected component, and then the union of them is the solutions for the whole graph.

Let $G = (U, V, E)$ be a bipartite graph such that $|U \cup V| = n$. For every vertex of G , we assign a variable x_i . Then, the Minimum Vertex-Edge Problem can be formulated by an $(0, 1)$ -linear program as follows:

$$\begin{aligned} & \min \sum_{i=1}^n x_i \\ & \begin{cases} A^T X = I \\ x_i \in \{0, 1\} \end{cases} \end{aligned}$$

where A is the edge(row)-vertex(column) incidence matrix of G , $X^T = (x_1, x_2, \dots, x_n)$ and $I^T = (1, 1, \dots, 1)$. Because G is an undirected bipartite graph, A must be a totally unimodular matrix. So, the above programming can be solved simply by solving its relaxed linear programming, which can be done in polynomial time. So, the Vertex-Edge Problem and its corresponding minimum problem can be solved easily.

3 The Edge-Vertex Problem

Unlike the Vertex-Edge Problem, the Edge-Vertex Problem is not that easy to solve. At first, it is easy to see that the Edge-Vertex Problem has the following relation with the ODD SET Problem [5]: for a graph $G = (V, E)$, if we take the set of its edges as the set $R = E = \{e_1, e_2, \dots, e_m\}$, the set of its vertices as the set $B = V = \{v_1, v_2, \dots, v_n\}$ and e_i is adjacent to v_j if and only if v_j is an end-vertex of e_i , then the Edge-Vertex Problem for G is equivalent to the ODD SET Problem for the red/blue bipartite graph $G' = (R, B, E^*)$. But, this does not seem to help. In this section, we will find all the graphs that have a solution to the Edge-Vertex Problem, i.e., G has even number of vertices. If G has solutions, we give an algorithm to find a solution for G . Some properties of the solutions and minimum solutions are discussed. We get that a tree with even order has a unique solution to the Edge-Vertex Problem, and therefore a unique optimal solution to the Minimum Edge-Vertex Problem. Finally, by polynomially transforming the Minimum Edge-Vertex Problem to the Minimum Weight Perfect Matching Problem, we show that the Minimum Edge-Vertex Problem for general graphs can be solved in polynomial time.

Definition 3.1. For a graph G , if a spanning subgraph G' of G has the property that the degree of every vertex of G' is odd in G' , then G' is called an *odd degree spanning subgraph*.

Theorem 3.1. G has a solution to the Edge-Vertex Problem if and only if G contains an odd degree spanning subgraph G' .

Proof. If G has a solution, then for all $v \in V(G)$, the number of edges in the solution incident to the vertex v is odd. So, the subgraph G' induced by the edges in the solution is an odd degree spanning subgraph. The other round is obvious. The proof is complete. ■

The graphs that have a solution to the Edge-Vertex Problem are given as follows.

Theorem 3.2. A graph G has a solution to the Edge-Vertex Problem if and only if G has even number of vertices, i.e., G is of even order.

Proof. If G has a solution to the Edge-Vertex Problem, then from the above result, we know that G contains a odd degree spanning subgraph G' . Since the number of odd degree vertices in any graph is even, we know that G' has even number of vertices. Since G' is a spanning subgraph of G , so, G has even number of vertices.

Conversely, if G has even number of vertices, we will show that G has a solution to the Edge-Vertex Problem. Since any solution of a spanning tree of G is also a solution of G , it is sufficient to show that any tree T of even order has a solution to the Edge-Vertex Problem. By induction on $|T| = n$. If $|T| = 2$, the conclusion is obvious. Suppose that the conclusion is true for all even order trees with the number of vertices less than n . Then, for an even order tree T with $|T| = n$, we can choose a vertex v from T such that all but at most one neighbors of v are not leaves of T . Such a vertex v always exists by observing that the vertices next to the end vertices of a longest path of T have the required property. We then distinguish the following two cases:

Case 1. If all the neighbors of v are leaves, then $T = K_{1,n-1}$. It is easy to see that all the $n - 1$ edges of $K_{1,n-1}$ consist of a solution.

Case 2. Exactly one neighbor of v is not a leaf of T .

Subcase 2.1 Suppose the number of leaves adjacent to v is odd, say $l_1, l_2, \dots, l_{2k+1}$. Let T' be a subtree of T obtained by deleting v and the leaves $l_1, l_2, \dots, l_{2k+1}$ from T . It is clear that T' is also an even order tree since $|T'| = n - (2k + 1) - 1 = n - 2(k + 2)$. Then, from the induction hypothesis, T' contains a solution E' . Therefore, $E' \cup \{vl_1, vl_2, \dots, vl_{2k+1}\}$ is a solution of T .

Subcase 2.2 Suppose the number of leaves adjacent to v is even, say l_1, l_2, \dots, l_{2k} . Let T' be a subtree of T obtained by deleting the leaves l_1, l_2, \dots, l_{2k} from T . It is clear that T' is also a tree of even order with $|T'| = n - 2k$, and therefore contains a solution E' by induction hypothesis. We then obtain a solution $E' \cup \{vl_1, vl_2, \dots, vl_{2k}\}$ of T . The proof is complete. ■

Actually, the proof of the above theorem suggests an $O(n^2)$ or $O(|E(G)|)$ time algorithm for finding a solution to the Edge-Vertex Problem for a given graph with even order, which is stated as follows:

Algorithm to the Edge-Vertex Problem: First, find any spanning tree T of G . Then, find a solution to the Edge-Vertex Problem in T , by the method mentioned in the above

proof. All these can be done by the well-known BFS or DFS method.

Remark 3.1. The complexity of the above algorithm for a tree is obviously linear time, since each edge of a tree is scanned at most once.

Before giving the algorithm for finding an optimal solution to the Minimum Edge-Vertex Problem for a general graph, we show some properties on optimal solutions.

Theorem 3.3. If G is a connected graph with even number n of vertices, then any optimal solution S_{opt} to the Minimum Edge-Vertex Problem for G satisfies that $n/2 \leq |S_{opt}| \leq n - 1$. Both the lower bound and the upper bound are best possible.

Proof. First, find a spanning tree T of G , and then we can get a solution to the Edge-Vertex Problem for G from such a solution for T . So, $|S_{opt}| \leq n - 1$. Since every vertex of G must be covered by S_{opt} and every edge covers at most two vertices, we get that $n/2 \leq |S_{opt}|$. To see that the lower bound is best possible, we consider a graph G with a perfect matching M . Then, M is an optimal solution to the Minimum Edge-Vertex Problem for G and $|M| = n/2$. So, the lower bound can be reached by many graphs. To see that the upper bound is best possible, we consider the graph $G = K_{1,n-1}$. It is not hard to see that the unique solution to the Edge-vertex Problem for the graph G is composed of all the edges of G . So, the upper bound can be reached by the graph $G = K_{1,n-1}$. ■

Actually, we can construct many other examples, as long as the graph is a tree such that its every vertex has an odd degree.

The following result is immediate.

Theorem 3.4. For any graph G with even order n , there is an $O(n^2)$ time algorithm for the Minimum Edge-Vertex Problem of G that produces a $2(1 - 1/n)$ -approximation solution.

Proof. First, find a spanning tree T of G , and then find a solution S of G from T . Since each of them uses at most $O(n^2)$ time, the total complexity is $O(n^2)$. Since the solution S satisfies that

$$n/2 \leq |S_{opt}| \leq |S| \leq n - 1,$$

we have that

$$|S|/|S_{opt}| \leq (n - 1)/(n/2) = 2(1 - 1/n),$$

i.e., S is a $2(1 - 1/n)$ -approximation solution. ■

Theorem 3.5. The subgraph G' induced by any optimal solution to the Minimum Edge-Vertex Problem does not contain any cycle. Moreover, each connected component of G' induces a tree in G .

Proof. Since every vertex of G must be covered by the solution, G' must be a spanning subgraph of G . Next, we want to show that there is no cycle in G' . If there is a cycle C in

G' , then the edges of C must belong to the optimal solution. On the other hand, obviously $E(G') - E(C)$ is a solution to the Edge-Vertex Problem for G , which contradicts to that $E(G')$ is an optimal solution to the Minimum Edge-Vertex Problem for G .

For the second statement of the theorem, suppose that a connected component H of G' induces a subgraph that has a cycle. Since G' does not have any cycle and nor does H , there must be an edge e of G such that there is a cycle C in $H + e$. Then, $(E(G') - E(C)) \cup \{e\}$ is a solution to the Edge-Vertex problem for G with fewer edges than $E(G')$ since G is simple and so $|E(C)| \geq 3$, a contradiction to that $E(G')$ is optimal. The proof is complete. ■

From the above result, we get that the Minimum Edge-Vertex Problem is equivalent to the following problem:

The Problem of Odd Degree Induced Spanning Forest with Minimum Number of Edges: In a given graph G , find an induced spanning forest with Minimum number of edges such that each of its vertex has an odd degree.

From the relation that if the number of components in a forest is r then the number of edges in the forest is $n - r$, we get that the above problem is equivalent to the following problem:

The Problem of Odd Degree Induced Spanning Forest with Maximal Number of Components: In a given graph G , find an induced spanning forest with maximal number of components such that each of its vertex has an odd degree. Or, in other words, partition the set of vertices of G into as many parts as possible such that each part induces an odd degree tree in G .

Both the above two problems are interesting optimization problems. But, it seems there is no help to solving our original Minimum Edge-Vertex Problem. We will show that the Minimum Edge-Vertex Problem can be solved in polynomial time. As a consequence, the above two problems can also be solved in polynomial time.

The following facts are obvious: The order of every component in the spanning forest of G induced by an optimal solution is even, and the number of components is at most $n/2$. The upper bound is best possible, which can be reached by any graph with perfect matchings. So, if a graph G has a perfect matching, then a solution S is optimal if and only if S is a perfect matching of G .

There are many classes of graphs whose optimal solutions are exactly their perfect matchings, for examples, 2-edge-connected and 3-regular graphs have perfect matchings, k -regular bipartite graphs with $k > 0$ have perfect matchings, see [2]. Next we find the following surprising result whose proof is very simple. By *claw-free* in a graph G , we mean that G does not contain an induced $K_{1,3}$ as a subgraph, and we simply call the graph a *claw-free graph*.

Theorem 3.6. A claw-free graph G has a perfect matching if and only if G is of even order.

Proof. If G has a perfect matching, then obviously G is of even order. Conversely, if G is of even order, then G has a solution to the Edge-Vertex Problem. Suppose S is an optimal solution to the Minimum Edge-Vertex Problem for G . Then, each component of the subgraph of G induced by S is an induced tree in G . Since G is claw-free, we claim that each such component is a K_2 . Otherwise, some such component has a path of length 2, say uvw . Since v is of odd degree in the induced subgraph, there must be a vertex x different from u and w such that vx is an edge in S . Then, from Theorem 3.5, $\{u, v, w, x\}$ must induce a claw of G , a contradiction. ■

Remark 3.2. Note that the line graph of any graph is claw-free. So, a line graph with even order has perfect matchings, and therefore the optimal solutions to the Minimum Edge-Vertex Problem for a line graph are exactly the perfect matchings of the line graph.

The relation between the optimal solutions and the perfect matchings in a graph with perfect matchings suggests us the following Berge-type results, the proofs are omitted.

Theorem 3.7. For any two solutions S and S' to the Edge-Vertex Problem for a graph G , the symmetric difference $S \oplus S'$ induces an Eulerian subgraph of G .

Theorem 3.8. Let S be a solution to the Edge-Vertex Problem for a graph G . Then, S is optimal to the corresponding minimum problem if and only if for any cycle C of G we have that $|S \cap C| \leq |\bar{S} \cap C|$, where $\bar{S} = E(G) - S$.

As a consequence, we get the following result.

Theorem 3.9. Suppose that T is a tree with even number of vertices. Then, T has a unique solution to the Edge-Vertex Problem, and therefore a unique optimal solution to the Minimum Edge-Vertex Problem.

Proof. If H and K are two different solutions for T , then $H \oplus K$ is a nonempty Eulerian subgraph of T . So, there is a cycle in the tree T , a contradiction. The proof is complete. ■

From Theorem 3.9, we can get a unique solution from a spanning tree T of a connected G with even vertices. However, there is no one to one correspondence between the solutions and the spanning trees of a graph. Two different spanning trees may give the same solution for G .

All the above discussions help us to understand the structures of a (optimal) solution. Next we give the main result of this section.

Theorem 3.10. The Minimum Edge-Vertex Problem can be solved in polynomial time for general graphs G .

Proof. It is easy to check if a graph G has a solution to the Edge-Vertex Problem can be done in $O(n)$, simply by checking the order of G . To show that to find an optimal solution to the Minimum Edge-Vertex Problem can be done in polynomial time, we will polynomially transform the problem for a given graph G to the Minimum Weight Perfect

Matching Problem for another weighted graph G^* . Then, from [9] the later problem can be solved in polynomial time, and so can be done for the former problem. The proof is divided into the following steps.

Step 1. Construct the graph G^* from G .

Suppose that we are given a graph $G = (V(G), E(G))$ with even order such that $V(G) = \{v_1, v_2, \dots, v_n\}$. From our Algorithm to the Edge-Vertex Problem, we can obtain a solution S to the Edge-Vertex Problem for G in $O(n^2)$ time. Consider the subgraph G_0 of G induced by the S . We know that G_0 is an odd degree spanning subgraph of G . For every vertex v of G , assume that the neighborhood of v_i in G is $N_i = \{u_1, u_2, \dots, u_d\}$, and the neighborhood of v_i in G_0 is $\{u_1, u_2, \dots, u_{2k-1}\}$, where $k \geq 1$ and $2k-1 \leq d$. We distinguish the following cases.

Case 1.1. d is odd, say $d = 2r - 1$.

Obviously, $r \geq k$. Let $V_i^1 = \{v_i^{11}, v_i^{12}, \dots, v_i^{1d}\}$ and $V_i^2 = \{v_i^{21}, v_i^{22}, \dots, v_i^{2d}\}$ be two new sets of vertices corresponding to the vertex v_i . Completely connect the vertices in V_i^1 to form a complete graph with d vertices, and assign each of the edges with weight 0. Then, connect v_i^{1j} and v_i^{2j} with an edge for every $j = 1, 2, \dots, d$, and assign each of these edges with weight 1. Finally, connect v_i^{2j} and u_j with an edge for every $j = 1, 2, \dots, d$, and assign each of these edges with weight 0. The edge $v_i^{2j}u_j$ can be imagined to be equal to the edge v_iu_j for every $j = 1, 2, \dots, d$.

Case 1.2. d is even, say $d = 2r$.

Obviously, $r \geq k$. Let V_i^1 and V_i^2 be the same sets as in Case 1.1. The edges among the vertices in the three sets V_i^1 , V_i^2 and N_i are also the same as in Case 1.1. However, we add one more new vertex w_i for v_i . Then, connect w_i and v_i^{1j} with an edge for every $j = 1, 2, \dots, d$, and assign each of these edge with weight 0. So, $V_i^1 \cup \{w_i\}$ forms a complete graph with $d + 1$ vertices.

Let $W = \{w_i | d_G(v_i) = 0 \pmod{2}\}$. Then, the new graph G^* has $W \cup (\bigcup_{i=1}^n V_i^1) \cup (\bigcup_{i=1}^n V_i^2)$ as its vertex set. The edge set of G^* consists of all the edges in Cases 1.1 and 1.2 with a small change for the kind of edges $v_i^{2j}u_j$, because then u_j is also blown up into two sets of vertices (plus one more vertex if $d_G(u_j)$ is even). In general, we call the vertices v_i^{2j} for $j = 1, 2, \dots, d$ the representatives of the vertex v_i in G^* . If v_iv_j is an edge of G , then v_j is a neighbor of v_i and v_i is a neighbor of v_j . Suppose that v_j is the p -th neighbor of v_i and v_i is the q -th neighbor of v_j . Then we use the edge $v_i^{2p}v_j^{2q}$ in G^* to substitute the edge $v_i^{2p}v_j$ as in Cases 1.1 and 1.2, which represents the edge v_iv_j of G . Now the construction of the new graph G^* is ready. Obviously, the construction can be done in polynomial time in n .

Step 2. Show that G^* has a perfect matching.

We construct a perfect matching M with the following three parts:

Part 2.1. Recall that we have an odd spanning subgraph G_0 . For every edge $v_i v_j$ in $E(G) - E(G_0)$, we order that the corresponding edge $v_i^{2p} v_j^{2q}$ in G^* belongs to M .

Note that these matched even number of vertices in V_i^2 if v_i has an odd degree in G , and odd number of vertices in V_i^2 if v_i has an even degree in G , for every $i = 1, 2, \dots, n$.

Part 2.2. For each odd degree vertex v_i , we match the even number of vertices v_i^{1j} in V_i^1 corresponding to the matched vertices v_i^{2j} in V_i^2 by any perfect matching among them. Then, match the remaining unmatched vertices in V_i^1 and V_i^2 by $v_i^{1j} v_i^{2j}$.

Note that all the vertices blown up at v_i are matched now.

Part 2.3. For each even degree vertex v_i , we match the odd number of vertices v_i^{1j} in V_i^1 corresponding to the matched vertices v_i^{2j} in V_i^2 by a perfect matching among them plus the extra one vertex w_i . The rest can be done in the same way as in Part 2.2.

Obviously, the above constructed is a perfect matching for the graph G^* .

Step 3. Show that every solution to the Edge-Vertex Problem for the original graph G , i.e., the set of edges of every odd degree spanning subgraph of G corresponds to a perfect matching of G^* .

The corresponding perfect matching G^* can be constructed in the same way as in Step 2.

Step 4. Denote the set of edges $v_i^{2p} v_j^{2q}$ in G^* corresponding to the edges $v_i v_j$ in G by $E^*(G)$. Then there is a one to one correspondence φ from $E^*(G)$ onto $E(G)$. Show that every perfect matching M of G^* corresponds, under $\varphi|_{\overline{M} \cap E^*(G)}$, to the set of edges of an odd degree spanning subgraph of G , i.e., a solution to the Edge-Vertex Problem for the graph G .

We only need to show that M always matches odd number of edges between V_i^1 and V_i^2 for any vertex v_i , since then for each matched edge between V_i^1 and V_i^2 there is exactly one unmatched edge between V_i^2 and N_i , which corresponds to the edges in an odd degree spanning subgraph of G . We distinguish two cases.

Case 4.1. v_i is an odd degree vertex of G .

Since M is a perfect matching, all the vertices in V_i^1 must be matched. In any case, there are even number of vertices matched among V_i^1 . Then, the rest odd number of vertices in V_i^1 unmatched among V_i^1 must matched with odd number of vertices in V_i^2 . Our required is proved.

Case 4.2. v_i is an even degree vertex in G .

Since M is a perfect matching, all the vertices in $V_i^1 \cup \{w_i\}$ must be matched. Note that the vertex w_i must be matched with a vertex in V_i^1 . In any case, the rest odd number

of vertices in V_i^1 can match even number of them among themselves. So, the number of vertices of V_i^1 matched with the vertices in V_i^2 is odd. Again, the required is proved.

Step 5. Note that G^* is a weighted graph with perfect matchings. Each of the edges between V_i^1 and V_i^2 is assigned weight 1, while each of the other edges is assigned weight 0. Show that the weight of a minimum weight perfect matching M of G^* is equal to the number of edges of an optimal solution to the Minimum Edge-Vertex Problem for G , or the number of edges of an odd degree spanning subgraph of G with minimum number of edges.

From the proof of Step 4, a minimum weight perfect matching M of G^* corresponds to an odd degree spanning subgraph of G with the number of edges equal to the weight of M . So, the weight of a minimum weight perfect matching is at least the number of edges of an odd degree spanning subgraph of G with Minimum number of edges. Conversely, from Step 2, an odd degree spanning subgraph G_0 of G with Minimum number of edges can be used to construct a perfect matching M of G^* . It is easy to see that the number of edges in G_0 is equal to the weight of M . So, the number of edges in an odd degree spanning subgraph of G with Minimum number of edges is at least the weight of a minimum weight perfect matching of G^* . Therefore, the minimum weight and the Minimum number must be equal.

Up to now, we have proved that the Minimum Edge-Vertex Problem for a given graph G is equivalent to the Minimum Weight Perfect Matching Problem for a new graph G^* polynomially constructed from G . From [9] we know that the Minimum Weight Perfect Matching Problem can be solved in polynomial time. So, the Minimum Edge-Vertex Problem can also be solved in polynomial time. The proof is now complete. ■

Remark 3.3. The above theorem tells us that three other problems can also be solved in polynomial time, i.e., the Problem of Odd Degree Induced Spanning Forest with Minimum Number of Edges, the Problem of Odd Degree Induced Spanning Forest with Maximal Number of Components, and the Odd Set Problem [5] for a red/blue bipartite graph (R, B, E) such that each vertex in R has degree 2.

4 Concluding Remarks

As we mentioned before, it is easy to see that the Edge-Edge Problem for a given graph G is equivalent to the Vertex-Vertex Problem for the line graph $L(G)$ of G . So, the existence of solutions to the Edge-Edge Problem for general graphs is solved, and to find solutions can also be done in polynomial time. However, for the Minimum Edge-Edge Problem, it seems no help for finding an optimal solution efficiently for this special kind of graphs, i.e., line graphs, which have many particular properties, for example, claw-free. We shall leave the minimum (optimization) problem for further study.

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